A Differential Game-Theoretic Approach to Sliding Mode Control

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Abstract

In this paper, the concepts of sliding mode control and game theory are combined to bring a new approach to the design of sliding controllers. In particular, we examine the problem of tracking with imprecise system models and present a game-theoretic interpretation of the problem.

1. Introduction

In the control of nonlinear systems with uncertainty, a simple type of control is sliding mode control [1, 2]. Motivated by the idea that it is much easier to control a first order system than it is to control a general n-th order system, sliding mode control transforms the n-th order problem to an equivalent first order problem. This simplified first order problem can then be solved by choosing a control law that satisfies the so-called sliding condition. The classic approach to the design of sliding mode control consists of two stages [3]: First, for the transformed problem, a feedback control law that can achieve perfect performance is selected. Then, this control law is modified in order to compensate for the neglected dynamics and disturbances.

In this paper, the concepts of sliding mode control and game theory are combined to bring a new approach to the design of sliding controllers. In particular, we examine the problem of tracking with imprecise system models and present a game-theoretic interpretation of the problem. In this approach, one views the system as a differential Nash game being played between two players, where the system dynamics are being dictated by the sliding surface dynamics. The equivalence of this formulation with the classical approach establishes the validity of such an interpretation. However, such an approach brings a fundamentally new perspective on the realization of sliding mode control systems. In section 2, the problem of tracking with imprecise system models is discussed. A brief overview of the classical approach to sliding mode control is provided in section 3. Following, game-theoretic formulation is presented. The paper concludes with a brief summary.

2. Tracking With Imprecise System Models

Consider the single input dynamic system:

\[ x^{(n)} = f(x,t) + b(x,t)u \]

where \( x \) is the scalar output, \( u \) is the scalar control input. Let us assume the function \( b(x,t) \) to be identically equal to 1. In particular, let us consider the following system dynamics:

\[ \ddot{x} = f(x,t) + u \]  \hspace{1cm} (1)

\( f(x,t) \) is not known exactly. It is estimated to be \( \hat{f}(x,t) \), where the estimation error is known to be bounded by a known continuous function \( F(x,t) \) as:

\[ |f(x,t) - \hat{f}(x,t)| \leq F(x,t) \]

The control problem is to design a provably correct control \( u \) which ensures that the state \( x = [x, \dot{x}, ..., x^{n-1}] \) tracks a specific time varying state \( x_d = [x_d, \dot{x}_d, ..., x_{d}^{n-1}] \) with this type of modeling imprecision.

3. Sliding Mode Control: Classical Approach

In sliding mode control [4], one can define a time varying surface \( S(t) \) by the scalar equation \( s(x,t) = 0 \), where

\[ s(x,t) = \left( \frac{d}{dt} + \lambda(t) \right)^{n-1} \dot{x} \]  \hspace{1cm} (2)

where \( \dot{x} = x - x_d \). The tracking problem is transformed into that of remaining on the surface \( S(t) \) for all \( t > 0 \). Instead of tracking the n dimensional vector \( x_d \), one needs only to keep the scalar quantity \( s \) at zero, i.e. a first order stabilization problem in \( s \). It has been shown that the simplified problem can be solved by choosing the control law \( u \) which satisfies the following condition:

\[ \frac{1}{\eta} \frac{ds^2}{dt} \leq -\eta |s| \]

\( \eta \) is a strictly positive constant.

This condition - referred to as the sliding condition - actually states that the squared distance to the surface, as measured by \( s^2 \), decreases along all system trajectories and consequently implies that some disturbances or dynamic uncertainties can be tolerated while still keeping the surface an invariant set. The structure of the controller is composed of a nominal part, similar to a feedback linearizing or inverse control law, and of an additional discontinuous term aimed at dealing with modeling uncertainty [4]. Thus, the design procedure consists of two steps.

First, a feedback control law is selected by constructing Filippov's equivalent dynamics via equating the time derivative of \( s \) to zero. Solving this equation formally for the control input, one obtains an expression for \( u \) called the equivalent control \( u_{eq} \). The equivalent control can be interpreted as the continuous control law if the dynamics were exactly known. If the dynamical system is described by eq. 1, the time derivative of the sliding surface is:

\[ s = \dot{x} + \lambda(t)\ddot{x} \]

\[ \dot{s} = \ddot{x} - \ddot{x}_d + \lambda(t)\ddot{x} + \lambda(t)\ddot{x} \]

Requiring \( \dot{s} = 0 \), \( u_{eq} \) is derived as:

\[ u_{eq} = \ddot{x}_d + \lambda(t)\ddot{x} - \lambda(t)\ddot{x} - f(x,t) \]

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Since \( f(x, t) \) is unknown, using the estimate of \( f \), an estimate of \( u_0 \) is obtained as:

\[
\dot{u}_0 = \dot{x}_d - \lambda(t)\dot{x} - \lambda(t)\dot{z} - f(x, t)
\]

Secondly, in order to compensate for the uncertainty in the system dynamics, a discontinuous term is added to this control and the final control law is thus obtained.

\[
u = \dot{x}_d - \lambda(t)\dot{x} - \lambda(t)\dot{z} - f(x, t) + k(x, t)\text{sgn}(s)
\]

\( k(x, t) \) is the gain whose structure will be determined in the sequel as to make this control law satisfy the sliding condition. Following a reasoning similar to that in [4] and [5], let us choose \( V = s^2/2 \) as the Lyapunov function and take its derivative with respect to time:

\[
\dot{V} = s \dot{s} = (f(x, t) - f(x, t))s - k(x, t) | s |
\]

Let us choose \( k(x, t) = F(x, t) + \eta \) where \( \eta > 0 \). Then \( k(x, t) > 0 \) and with this choice, even if \( f - f \) is positive, its magnitude will always be less than \( k(x, t) | s | \). Consequently, \( V < 0 \). From which the following inequality follows:

\[
\frac{1}{2} \frac{ds^2}{dt} \leq -\eta | s |
\]

As will be recalled, this is the sliding condition.

### 4. Differential Games

Continuous-time infinite dynamical games - also known as differential games - constitute a class of decision problems wherein the evolution of the state is described by a differential equation and the players act throughout a time interval [6](pp:210). One can formulate these games by defining the following basic components:

1. A set of players \( P_i \) where \( i = 1, \ldots, N \).
2. A time interval \([0, T]\) which describes the duration of the game.
3. An infinite set \( S_0 \) called the trajectory space of the game, where \( x \in S_0 \) denotes a state vector.
4. An infinite set \( U \) called the control action space of player \( P_i \).
5. A differential equation \( \dot{x} = f(t, x, u_1, \ldots, u_N) \) with an initial \( x(0) = x_0 \) whose solution describes the state trajectory of the game.
6. An information field.
7. A set of strategy spaces.
8. A cost functional \( L_0 \) defined for each player, each of the form \( L_0(u_1, \ldots, u_N) \)

A differential game is not well-defined unless an information structure is prescribed to the game. The reader is referred to [6] for a detailed explanation of these concepts. In the sequel to follow, let us consider memoryless perfect state (feedback) information structure which provides the players with only the current value of the state at each stage of the game (and of course the initial state).

If the players adopt the noncooperative Nash equilibrium solution concept, each player is faced with an optimal control problem with the strategies of the remaining players taken to be fixed at their equilibrium values. Hence, in order to derive the Nash equilibria, one has to utilize tools of optimal control theory - in particular the minimum principle. Generalising the optimal control problem to a game-theoretic setting, a theorem [6](pp:296) provides a set of necessary conditions for any feedback Nash equilibria to satisfy under the feedback information structure. Define \( u = [u_1^T, \ldots, u_N^T]^T \) as the aggregate control vector and \( p_i \) to be the costate vector associated with player \( P_i \). Applying the conditions of this theorem here, an Hamiltonian \( H_i \) is defined for each player \( P_i, i = 1, \ldots, N \) as:

\[
H_i(t, p_i, x, u) = g_i(t, x, u) + p_i^T f(t, x, u)
\]

The state and the costate equations become:

\[
p_i^T = -\frac{\partial H_i}{\partial x} \quad x^* = \frac{\partial H_i}{\partial p_i}
\]

The optimal \( u_i^* \) can then be found by:

\[
u_i^* = \arg\min_{u_i} H_i(t, p_i^*, x^*, u_1, u_1^*, \ldots, u_i^*, \ldots, u_N^*)
\]

with the boundary conditions

\[
p_i(T) = \frac{\partial H_i(T, x^*)}{\partial x} \quad i = 1, \ldots, N
\]

\[
x(0) = x_0
\]

### 5. Differential Nash Game Interpretation of SMC

Interestingly, a differential game-theoretic interpretation of the problem yields a control law identical to the classical approach. In this section, we will first present a differential game formulation of sliding mode control and then show the equivalence between the two.

#### 5.1 Differential Feedback Nash Game Formulation

In order to develop a differential game-theoretic formulation, let us make the following change of variables in the functions \( f(x, t), \dot{f}(x, t), k(x, t) \):

\[
\begin{align*}
\hat{x} & = x + \lambda(t)\dot{x} + \lambda(t)\dot{z} \\
\hat{t} & = t - \lambda(t)\dot{z} + u_1 \\
\end{align*}
\]

Then, consider the following two-player differential Nash game where the system dynamics are:

\[
\begin{align*}
\dot{\hat{x}} = F^*(\hat{z}, t) + k^*(\hat{z}, t)u_1 \\
\dot{\hat{z}} = -\lambda(t)\dot{\hat{z}} + u_2 \\
\end{align*}
\]

where \( k^*(\hat{z}, t) > 0 \) and \( u_i, i = 1, 2 \) represents the decision variable of \( P_i \). Assume \( u_1 \in [-1, 1] \). The respective objective functions \( L_1 \) and \( L_2 \) are defined as:

\[
\begin{align*}
L_1(u_1, u_2; T) & = \int_0^T -s \dot{t} dt \\
L_2(u_1, u_2; T) & = \int_0^T (\lambda(t)u_1 - \hat{u}_2) \dot{t} dt \\
\end{align*}
\]

Let us define the costate vectors as \( p_i^T = [p_{i1}, p_{i2}] \). The Hamiltonians \( H_i \) for each player \( i = 1, 2 \) is then defined as the following:
5.2 Equivalence with Classical Interpretation

It turns out the differential feedback Nash game interpretation leads to a control which is equivalent to that of the classical approach. In order to demonstrate this, let us first replace the optimal controls \( u_1^* \) and \( u_2^* \) in the state equations as follows:

\[
\dot{s} = F^*(\tilde{s}, t) + k^*(\tilde{s}, t)(-\text{sgn}(s)) \tag{4}
\]

\[
\dot{\tilde{s}} = -\lambda(t)\tilde{s} + s \tag{5}
\]

Let us manipulate eq. 5 and use eq. 4 as follows:

\[
\dot{s} + \lambda(t)s = s
\]

\[
\dot{s} + \lambda \tilde{s} + \lambda \tilde{s} = \dot{s}
\]

\[
F^*(\tilde{s}, t) - k^*(\tilde{s}, t)\text{sgn}(s) \tag{6}
\]

Using the definition of \( F^*(\tilde{s}, t) \) and noting that \( \tilde{s} = \tilde{s} - \tilde{s}_0 \), eq. 6 is remanipulated into the form:

\[
\tilde{s} - \tilde{s}_0 + \lambda \tilde{s} + \lambda \tilde{s} = F^*(\tilde{s}, t) - f(\tilde{s}, t) - k^*(\tilde{s}, t)\text{sgn}(s)
\]

Changing the variables back to the original ones:

\[
\tilde{s} = \tilde{s}_0 - \lambda \tilde{s} - \lambda \tilde{s} + f(x, t) - f(x, t) - k(x, t)\text{sgn}(s) \tag{7}
\]

If we define the RHS of eq. 7 to be \( f(x, t) + u \), one obtains the original system \( \dot{s} = f(x, t) + u \) with \( u \) as

\[
u = \tilde{s}_0 - \lambda \tilde{s} - \lambda \tilde{s} - f(x, t) - k(x, t)\text{sgn}(s)
\]

As will be recalled, this was the control law derived for SCM with the classical approach. Also, eq. 5 can be written in the form \( s = \dot{s} + \lambda(t)\tilde{s} \) which is the scalar variable defined in SMC.

5.3 Realization As a Differential Nash Game

In the game-theoretic framework, one views the system as a differential Nash game being played between two players as shown in Figure 1. At each stage of the game, each player computes an optimal value for its decision variable based on its Nash strategy for that stage of the game, broadcasts it to the other player, and then updates its decision based on the newly received information. Both the decision variables \( u_1 \) and \( u_2 \) are used in updating the game dynamics, namely:

\[
\dot{s} = F^*(\tilde{s}, t) + k^*(\tilde{s}, t)u_1
\]

\[
\dot{\tilde{s}} = -\lambda(t)\tilde{s} + u_2
\]

At the same time, the decision variable \( u_1 \) of player \( P_1 \) can be used in generating the control input to the plant as:

\[
u(u_1) = \tilde{s}_0 - \lambda \tilde{s} - \lambda \tilde{s} - f(x, t) + k(x, t)u_1
\]

This generated control is equivalent to that generated in the classical approach.

6. Conclusion

This paper has presented a new approach to the design of sliding controllers. In particular, the problem of tracking with imprecise system models is investigated and a game-theoretic interpretation of the problem is developed. In this approach, one views the system as a differential Nash game being played between two players, where the system dynamics are being dictated by the sliding mode dynamics. The validity of this approach is established by showing its equivalence with the classical approach. However, by viewing the problem as a differential game, this approach brings a fundamentally new perspective on realizing sliding mode control systems. As future work, we will be ex-
amining the tracking problems where (i) the control coefficient \( k(x, t) \) is not taken to be one and (ii) the plant dynamics are of the form \( z^{(n)} = f(x, t) + k(x, t)u \) where the state \( x \) is an \( n \)-dimensional vector.

REFERENCES


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Fig. 1. Differential Nash Game Interpretation.